Written Homework 7

① $(1)v_1+(-1)v_2+(2)v_3=\begin{bmatrix}2\\3\\0\end{bmatrix}$. Basis, so one example of the basis could be
$v_1=\begin{bmatrix}0\\3\\1\end{bmatrix}, v_2=\begin{bmatrix}0\\0\\-1\end{bmatrix}, v_3=\begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow \boxed{B=\left\{\begin{bmatrix}0\\3\\1\end{bmatrix},\begin{bmatrix}0\\0\\-1\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}}$

Yes, there could be more examples because the vectors just need to satisfy the conditions of being linearly independent (to ensure they span $\mathbb{R}^3$) and having $v_1-v_2+2v_3=\begin{bmatrix}2\\3\\0\end{bmatrix}$, but other bases like $B=\left\{\begin{bmatrix}2\\-3\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ or $B=\left\{\begin{bmatrix}2\\0\\0\end{bmatrix},\begin{bmatrix}0\\-3\\2\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ work too.

② (a) In order for $\mathcal{B}_1$ to be a basis for $\mathbb{R}^3$, u_1, u_2 & u_3 must be linearly independent, meaning that they will form a basis for $\mathbb{R}^3$ in any order including $\mathcal{B}_2=\{u_3,u_1,u_2\}$. So $\mathcal{B}_2$ must be a basis for $\mathbb{R}^3$.

(b) $v_{\mathcal{B}_1}=\begin{bmatrix}1\\0\\0\end{bmatrix}$

Since u_1 is in the place of u_2 in the basis $\mathcal{B}_2$, $v_{\mathcal{B}_2}$ must $=\begin{bmatrix}0\\1\\0\end{bmatrix}$

This can also be verified computationally by:

$v=(1)u_1+(0)u_2+(0)u_3$

$v=u_1 \Rightarrow \boxed{v_{\mathcal{B}_2}=\begin{bmatrix}0\\1\\0\end{bmatrix}_{\mathcal{B}_2}}$

(c) $C\vec{v}_{\mathcal{B}_1}=\vec{v}_{\mathcal{B}_2} \Longrightarrow \boxed{C=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}}$

③ $\det(A)=(-1)^{11}(2\cdot3\cdot4)\left(\det\begin{bmatrix}5&6\\6&5\end{bmatrix}\right)(4\cdot3\cdot2\cdot1)$ } since matrix is upper triangular

$=-(24)(-11)(24)=6{,}336$

$\det(5A^3)=5^n(6{,}336)^3=5^{10}(6{,}336)^3\approx 2.484\times10^{18}$

④ (a) $c_{11}=(-1)^2\det(M_{11})=(-1)^2\det\begin{bmatrix}1&1\\-1&5\end{bmatrix}=6$

$c_{12}=(-1)^3\det(M_{12})=(-1)^3\det\begin{bmatrix}1&1\\3&5\end{bmatrix}=-2$

$c_{13}=(-1)^4\det(M_{13})=(-1)^4\det\begin{bmatrix}1&1\\3&-1\end{bmatrix}=-4$

$c_{21}=(-1)^3\det(M_{21})=(-1)^3\det\begin{bmatrix}0&2\\-1&5\end{bmatrix}=-2$

$c_{22}=(-1)^4\det(M_{22})=(-1)^4\det\begin{bmatrix}-2&2\\3&5\end{bmatrix}=-16$

$c_{23}=(-1)^6\det(M_{23})=(-1)^6\det\begin{bmatrix}-2&0\\3&-1\end{bmatrix}=-2$

$c_{31}=(-1)^4\det(M_{31})=(-1)^4\det\begin{bmatrix}0&2\\1&1\end{bmatrix}=-2$

$c_{32}=(-1)^5\det(M_{32})=(-1)^5\det\begin{bmatrix}-2&2\\1&1\end{bmatrix}=4$

$c_{33}=(-1)^6\det(M_{33})=(-1)^6\det\begin{bmatrix}-2&0\\1&1\end{bmatrix}=-2$

$$\boxed{B=\begin{bmatrix}6&-2&-4\\-2&-16&-2\\-2&4&-2\end{bmatrix}}$$

(b) $AB^T=\begin{bmatrix}-2&0&2\\1&1&1\\3&-1&5\end{bmatrix}\begin{bmatrix}6&-2&-2\\-2&-16&4\\-4&-2&-2\end{bmatrix}=\boxed{\begin{bmatrix}-20&0&0\\0&-20&0\\0&0&-20\end{bmatrix}} \Rightarrow AB^T=\det(A)I$

⑤ (a) $\det(A) = \det\begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix} = -4 + 3 = -1$

$\det(B) = \det\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = 0 + 1 = 1$

(b) A nonzero determinant means that the linear transformation is invertible. Via the Big Theorem, A being invertible means that A is a basis for $\mathbb{R}^2$.

(c) $A^{-1} = \frac{1}{-4+3}\begin{bmatrix} 1 & -3 \\ 1 & -4 \end{bmatrix} = -\begin{bmatrix} 1 & -3 \\ 1 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 3 \\ -1 & 4 \end{bmatrix}}$

$B^{-1} = \frac{1}{0+1}\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}}$

(d) $AB = \begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} -3 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3c_1 + 5c_2 \\ -1c_1 + 2c_2 \end{bmatrix} \Rightarrow c_1 = 3,\ c_2 = 2$

Coordinates of the point (1, 1) in the basis formed by columns of AB = (3, 2)

(e) $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T(x) = Ax$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = AB\begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix} = (AB)^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{A = \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}}$

checking that it works:

$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Written Homework 7

(1) Find an example of a basis $\mathcal{B}$ of $\mathbb{R}^3$ such that $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$. Is there more than one such example?

(2) Let $\mathcal{B}_1 = \{u_1, u_2, u_3\}$ be a basis for $\mathbb{R}^3$. Let $\mathcal{B}_2 = \{u_3, u_1, u_2\}$.

(a) Justify that $\mathcal{B}_2 = \{u_3, u_1, u_2\}$ is a basis for $\mathbb{R}^3$.

(b) Let $\mathbf{v}_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}_1}$. What is $\mathbf{v}_{\mathcal{B}_2}$?

(c) What is the matrix C such that for any vector $\mathbf{v}$, $C\mathbf{v}_{\mathcal{B}_1} = \mathbf{v}_{\mathcal{B}_2}$?

(3) Compute $\det(5A^3)$ for the matrix A below.

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 5 & 6 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 5 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

(4) Recall that if A is an $n \times n$ matrix, then M_{ij} denotes the matrix obtained from A by deleting the ith row and jth column of A.

$$\text{Let } A = \begin{bmatrix} -2 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & -1 & 5 \end{bmatrix}.$$

(a) Compute all nine cofactors of A, as well as $\det(A)$. Let B be the 3×3 matrix containing the cofactors, with each entry multiplied by the appropriate $\pm$ sign. So the ij-entry of B is $(-1)^{i+j}\det(M_{ij})$.

(b) Compute AB^T. You should get a diagonal matrix with the same number in every diagonal entry, in other words, a multiple of the identity matrix. What multiple is it (in terms of A)?

(c) Fill in the blank (with a scalar) to make this equation true:

$$AB^T = (\ ?\ ) \cdot I, \quad \text{therefore} \quad A^{-1} = \frac{1}{(\ ?\ )} \cdot B^T.$$

(d) Compute the matrix $B^\top$ when A is the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and then use it to compute a formula for A^{-1}. Does this agree with the formula for A^{-1} given in the textbook?

The formula described above works for computing the inverse of any $n \times n$ matrix. This formula does not provide an *efficient algorithm (computer science terminology)* for computing inverses. Why? Compare it to the method of computing inverses using Gaussian elimination, and think about how many additions, subtractions, multiplications and divisions go into the two methods.

The cofactor formula given above is however important and useful for mathematical arguments.

(5) Let $A = \begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$.

(a) Compute $\det(A)$ and $\det(B)$.
(b) Using facts about determinants and without computing AB, argue that the columns of AB form a basis of $\mathbb{R}^2$.
(c) Compute A^{-1} and B^{-1}.
(d) Compute the coordinates of the point $(1, 1)$ in the basis formed by the columns of AB. Express your computation abstractly in terms of A and B before you use the specific A and B given above.
(e) What is the linear transformation T that will undo the change of basis in (c) bringing your answer in (c) back to $(1, 1)$? Write T with domain, codomain and matrix. Express your computation abstractly in terms of A and B before you use the specific A and B given above. Check that it works.